Algebraic Number Theory Exam

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Exercise 1

(i) Which of the following is integral over Z? Motivate your answer:

$$\alpha_1 = \sqrt{2} + \sqrt{3} \quad , \quad \alpha_2 = \frac{1}{\sqrt{2}} \quad , \quad \alpha_3 = \frac{1+i}{\sqrt{2}}$$

(ii) Compute $disc(\mathbb{Q}(\alpha))$ where $\alpha = \sqrt[4]{2}$

(iii) Is 30 a square modulo 101? (101 is prime)

(iv) Let K be a number field with $2s$ complex embeddings.
Let α be such that $K = \mathbb{Q}(\alpha)$. Show that

$$\mathrm{sing}(\mathrm{disc}(K)) = \mathrm{sign}(\mathrm{disc}(\mathbb{Z}[\alpha])) = (-1)^s$$

Exercise 2 Let $K = \mathbb{Q}(\sqrt{-31})$

(i) Prove that $N_{K/\mathbb{Q}}(\beta) \geq 4$ for every $\beta \in \mathcal{O}_K \setminus \mathcal{O}_K^\times$ with $\beta \neq 0$.

(ii) Compute a set of representatives for the class group

(iii) Prove that $Cl(K) \cong \mathbb{Z}/3\mathbb{Z}$ and give an explicit generator

Exercise 3
Let $K = \mathbb{Q}(\sqrt{-13}), \alpha = \sqrt{-13}$
The class group has $|Cl(K)| = 2$ and is generated by $\mathfrak{p} = (2, 1 + \sqrt{-13})$ which lies over 2, i.e. $2\mathcal{O}_K = \mathfrak{p}^2$

$$y^3 = x^2 + 13$$

Let $(x, y) \in \mathbb{Z}^2$ be a solution of this equation.

(i) Prove that x is even, y is odd. Prove that $13 \nmid xy$

(ii) Show that $(x + \alpha), (x - \alpha)\mathcal{O}_K$ are coprime

(iii) Show that $(x, y) \in \mathbb{Z}^2$ is a solution if and only if $x + \sqrt{-13} = \beta^3, N(\beta) = y$ for $\beta \in \mathcal{O}_K$

(iv) Find all integer solutions $(x, y) \in \mathbb{Z}^2$